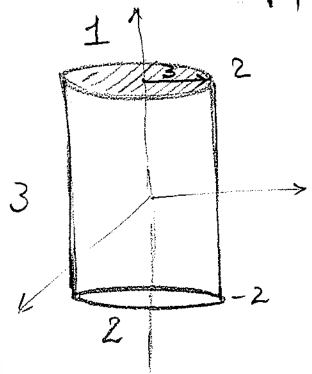
(تمرین ۲)

سؤال ۳ :



$$\vec{A} = 4x\hat{x} - 2y^2\hat{y} - 2z^2\hat{z}$$

$$\nabla\cdot\vec{A} = 4 - 4y - 4z = 4(1-y-z)$$

$$dv = r\,dr\,d\varphi\,dz$$

$$\int \nabla\cdot\vec{A}\,dv = \int 4(1 - r\sin\varphi - z)\,r\,dr\,d\varphi\,dz$$

$$= 4\int_0^3\int_0^{2\pi}\left\{z - r\sin\varphi\, z - \frac{z^2}{2}\right\}_{-2}^{2} r\,dr\,d\varphi = 16\int_0^3\int_0^{2\pi}(1 - r\sin\varphi)\,r\,dr\,d\varphi$$

$$= 16\int_0^3\left\{\varphi + r\cos\varphi\right\}_0^{2\pi} r\,dr = 16\int_0^3 2\pi\, r\,dr = 32\pi\cdot\frac{r^2}{2}\Big|_0^3 = \underline{144\pi} \quad *$$

up: $z=2$; $\vec{A} = 4x\hat{x} - 2y^2\hat{y} - 8\hat{z}$ $\quad \vec{ds} = r\,dr\,d\varphi\,\hat{z}$

$$\int_{up}\vec{A}\cdot\vec{ds} = -\int 8\,r\,dr\,d\varphi = -8(9\pi) = \underline{-72\pi} \quad (1)$$

bottom: $z=-2$; $\vec{A} = 4x\hat{x} - 2y^2\hat{y} - 8\hat{z}$ $\quad \vec{ds} = -r\,dr\,d\varphi\,\hat{z}$

$$\int_{bot}\vec{A}\cdot\vec{ds} = \int 8\,r\,dr\,d\varphi = 8(9\pi) = \underline{72\pi} \quad (2)$$

$$\boxed{\begin{array}{l}\hat{x}\cdot\hat{r} = \cos\varphi \\ \hat{y}\cdot\hat{r} = \sin\varphi\end{array}}$$

side: $r=3$; $\vec{A} = 4r\cos\varphi\,\hat{x} - 2r^2\sin^2\varphi\,\hat{y} - 2z^2\hat{z} = 12\cos\varphi\,\hat{x} - 18\sin^2\varphi\,\hat{y} - 2z^2\hat{z}$

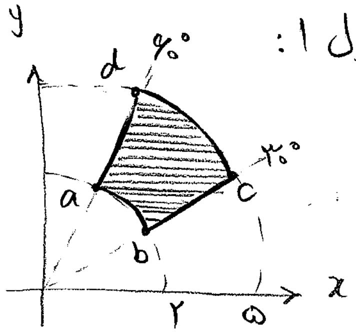
$$\vec{ds} = 3\,d\varphi\,dz\,\hat{r} \longrightarrow \int_{side}\vec{A}\cdot\vec{ds} = \int(12\cos\varphi\,\hat{x} - 18\sin^2\varphi\,\hat{y})\cdot\hat{r}\,3\,d\varphi\,dz$$

$$= 3\int_0^{2\pi}(12\cos^2\varphi - 18\sin^3\varphi)\,d\varphi\int_{-2}^{2}dz = 3\cdot(12\pi - 0)\cdot 4 = \underline{144\pi} \quad (3)$$

$1+2+3 = 144\pi$ $**$ $\quad$; $\quad *, ** \longrightarrow \int\nabla\cdot\vec{A}\,dv = \oint\vec{A}\cdot\vec{ds}$

(تمرین ۳)

سوال ۱:



$$\vec{A} = r\cos\varphi\,\hat{r} + \sin\varphi\,\hat{\varphi}$$

$$\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ r\cos\varphi & r\sin\varphi & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[0\,\hat{r} - 0\,r\hat{\varphi} + (\sin\varphi + r\sin\varphi)\hat{z}\right] = \frac{1}{r}(1+r)\sin\varphi\,\hat{z}$$

$$\int \nabla\times\vec{A}\cdot d\vec{s} = \int \frac{1}{r}(1+r)\sin\varphi\,\hat{z}\cdot r\,dr\,d\varphi\,\hat{z} = \int_{30}^{60}\sin\varphi\,d\varphi\int_{2}^{5}(1+r)\,dr$$

$$= -\cos\varphi\Big|_{30}^{60}\cdot\ r+\frac{r^2}{2}\Big|_{2}^{5} = \frac{\sqrt{3}-1}{2}\cdot\frac{27}{2} = \frac{27}{4}(\sqrt{3}-1) \quad *$$

ab: $r=2$; $\vec{A} = 2\cos\varphi\,\hat{r} + \sin\varphi\,\hat{\varphi}$ $\quad d\vec{l} = 2\,d\varphi\,\hat{\varphi}$

$$\int_{ab}\vec{A}\cdot d\vec{l} = \int_{60}^{30} 2\sin\varphi\,d\varphi = -2\cos\varphi\Big|_{60}^{30} = 1-\sqrt{3} \quad ①$$

bc: $\varphi = 30°$; $\vec{A} = \frac{\sqrt{3}}{2}r\,\hat{r} + \frac{1}{2}\hat{\varphi}$ $\quad d\vec{l} = dr\,\hat{r}$

$$\int_{bc}\vec{A}\cdot d\vec{l} = \int_{2}^{5}\frac{\sqrt{3}}{2}r\,dr = \frac{\sqrt{3}}{2}\frac{r^2}{2}\Big|_{2}^{5} = \frac{21\sqrt{3}}{4} \quad ②$$

cd: $r=5$; $\vec{A} = 5\cos\varphi\,\hat{r} + \sin\varphi\,\hat{\varphi}$ $\quad d\vec{l} = 5\,d\varphi\,\hat{\varphi}$

$$\int_{cd}\vec{A}\cdot d\vec{l} = \int_{30}^{60}5\sin\varphi\,d\varphi = -5\cos\varphi\Big|_{30}^{60} = \frac{5}{2}(\sqrt{3}-1) \quad ③$$

da: $\varphi = 60°$; $\vec{A} = \frac{r}{2}\hat{r} + \frac{\sqrt{3}}{2}\hat{\varphi}$ $\quad d\vec{l} = dr\,\hat{r}$

$$\int_{da}\vec{A}\cdot d\vec{l} = \int_{5}^{2}\frac{r}{2}\,dr = \frac{r^2}{4}\Big|_{5}^{2} = -\frac{21}{4} \quad ④$$

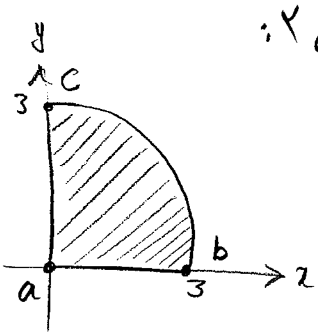
ادامه

$$1+2+3+4 = \frac{21}{4}(\sqrt{3}-1) + \frac{3}{2}(\sqrt{3}-1) = \frac{27}{4}(\sqrt{3}-1) \quad **$$

$$*,\ ** \longrightarrow \int \nabla\times\vec{A}\cdot d\vec{s} = \oint \vec{A}\cdot d\vec{l}$$

(تمرین ۳)

سؤال ۲:



$$\vec{A} = xy\,\hat{x} - 2x\,\hat{y}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = (-2-x)\,\hat{z}$$

$$\int \nabla \times \vec{A} \cdot d\vec{s} = -\int (x+2)\,\hat{z} \cdot r\,dr\,d\varphi\,\hat{z} = -\int (2+x)\,r\,dr\,d\varphi$$

$$= -\int_0^3 \int_0^{\pi/2} (2 + r\cos\varphi)\,r\,dr\,d\varphi = -\int_0^3 \Big\{ 2\varphi + r\sin\varphi \Big\}_0^{\pi/2} r\,dr$$

$$= -\int_0^3 (\pi + r)\,r\,dr = -\left(\frac{\pi r^2}{2} + \frac{r^3}{3}\right)_0^3 = -9\left(\frac{\pi}{2}+1\right) \quad *$$

روابطی که در حل از آنها استفاده شده است

$$\hat{x}\cdot\hat{\varphi} = -\sin\varphi$$
$$\hat{y}\cdot\hat{\varphi} = \cos\varphi$$
$$\int \sin^2\varphi\cos\varphi\,d\varphi = \frac{\sin^3\varphi}{3}$$
$$\int \cos^2\varphi\,d\varphi = \frac{1}{2}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)$$

ab: $\quad y=0 \;;\quad \vec{A} = -2x\,\hat{y} \qquad \vec{dl} = dx\,\hat{x}$

$$\int_{ab} \vec{A}\cdot\vec{dl} = \int_0^3 -2x\,\hat{y}\cdot dx\,\hat{x} = 0 \qquad (1)$$

bc: $\quad r = 3 \;;\quad \vec{A} = r^2\sin\varphi\cos\varphi\,\hat{x} - 2r\cos\varphi\,\hat{y} = 9\sin\varphi\cos\varphi\,\hat{x} - 6\cos\varphi\,\hat{y}$

$$\vec{dl} = 3\,d\varphi\,\hat{\varphi} \rightarrow \int_{bc} \vec{A}\cdot\vec{dl} = \int_0^{\pi/2} (9\sin\varphi\cos\varphi\,\hat{x} - 6\cos\varphi\,\hat{y}) \cdot 3\,d\varphi\,\hat{\varphi}$$

$$= \int_0^{\pi/2} (-27\sin^2\varphi\cos\varphi - 18\cos^2\varphi)\,d\varphi = -27\left(\frac{1}{3}\right) - 18\left(\frac{\pi}{4}\right) = -9\left(\frac{\pi}{2}+1\right) \qquad (2)$$

ca: $\quad x=0 \;;\quad \vec{A} = 0 \;,\quad \vec{dl} = dy\,\hat{y}$

روابط که در فوق را دانشجو می‌بایست بداند

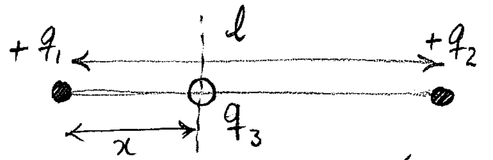
$$\int_{ca} \vec{A}\cdot\vec{dl} = 0 \qquad (3)$$

$$1+2+3 = -9\left(\frac{\pi}{2}+1\right) \quad **$$

$$*,** \longrightarrow \int \nabla\times\vec{A}\cdot\vec{ds} = \oint \vec{A}\cdot\vec{dl}$$

(تمرین ۳)

سؤال ۳ :



در کلاس بیان شد که بار سوم می بایست در بین دو بار دیگر قرار گیرد (چرا؟). آن را به فاصله‌ی x از بار $+q_1$ در نظر می گیریم:

$$|\vec{F}_{13}| = \frac{q_1 q_3}{4\pi\epsilon_0 x^2}$$

$$|\vec{F}_{23}| = \frac{q_2 q_3}{4\pi\epsilon_0 (l-x)^2}$$

شرط این که بدون حرکت باقی بماند این است که مجموع نیروهای وارد بر آن صفر باشد. یعنی $|\vec{F}_{13}| = |\vec{F}_{23}|$ (چرا؟)

$$\rightarrow \frac{q_1 q_3}{4\pi\epsilon_0 x^2} = \frac{q_2 q_3}{4\pi\epsilon_0 (l-x)^2} \longrightarrow \frac{q_1}{x^2} = \frac{q_2}{(l-x)^2}$$

داده شده $\quad q_1 = 1\,C \quad q_2 = 2\,C \quad l = 2\,m$

$$\rightarrow \frac{1}{x^2} = \frac{2}{(2-x)^2} \rightarrow (2-x)^2 = 2x^2 \rightarrow x^2 + 4x - 4 = 0$$

$$\Delta = 4+4 = 8 \qquad x_{1,2} = -2 \pm \sqrt{8} \longrightarrow \underset{>0}{x_1 = -2+\sqrt{8}} \ , \ \underset{<0}{x_2 = -2-\sqrt{8}}$$

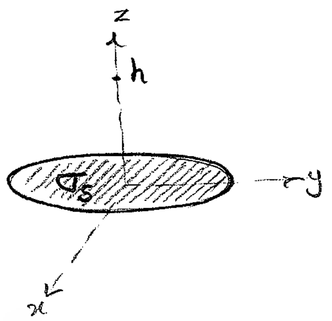
چون x فاصله است لذا مقدار x_1 قابل قبول خواهد بود.

$\longleftarrow$ هر باری به فاصله‌ی $(\sqrt{8}-2)$ از بار $+q_1$ و بین دو بار قرار گیرد بدون حرکت باقی خواهد ماند.

(تمرین ۴)

سؤال ۱:

$$\sigma_s = \cos\theta \xrightarrow{\text{در مختصات قطبی}} = \cos\varphi$$



چون بار تقارن ندارد نمی‌توان از فرمول حلقه استفاده کرد

و می‌بایست از فرمول میدان پاسخ را یافت:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_s\, ds'}{|\vec{R}-\vec{R}'|^3}(\vec{R}-\vec{R}')$$

$$\vec{R} = h\hat{z} \qquad \vec{R}' = r'\hat{r} \qquad ds' = r'dr'd\varphi' \qquad |\vec{R}-\vec{R}'| = (r'^2+h^2)^{1/2}$$

$$\rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{a} \frac{\cos\varphi'\, r'\,dr'\,d\varphi'}{(r'^2+h^2)^{3/2}}(h\hat{z}-r'\hat{r})$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{ h\int_0^a \frac{r'dr'}{(r'^2+h^2)^{3/2}} \cancelto{0}{\int_0^{2\pi}\cos\varphi' d\varphi'\,\hat{z}} \cdots \right.$$

$$\left. \cdots - \int_0^a \frac{r'^2dr'}{(r'^2+h^2)^{3/2}}\int_0^{2\pi}\cos\varphi'(\underbrace{\cos\varphi'\hat{x}+\sin\varphi'\hat{y}}_{\hat{r}})d\varphi' \right\}$$

می‌دانیم:

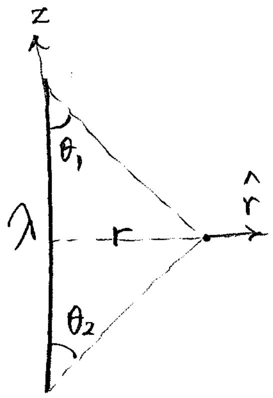
$$\int \frac{u^2}{(u^2+b^2)^{3/2}}du = \ln\left(\sqrt{u^2+b^2}+u\right) - \frac{u}{\sqrt{b^2+u^2}}$$

$$\int_0^{2\pi}\cos^2\varphi\, d\varphi = \pi \qquad\qquad \int_0^{2\pi}\sin\varphi\cdot\cos\varphi\, d\varphi = 0$$

$$\rightarrow \vec{E} = -\frac{1}{4\varepsilon_0}\cdot\left[\ln\left(\sqrt{a^2+h^2}+a\right) - \frac{a}{\sqrt{a^2+h^2}} - \ln(|h|)\right]\hat{x}$$

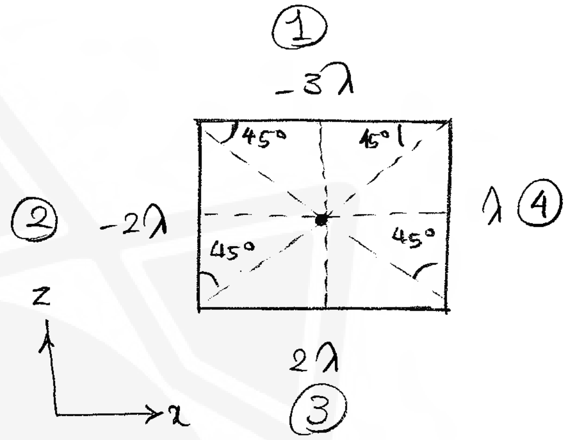
(تمرین ۴)

سؤال ۴:



$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0 r}\left\{\left(\cos\theta_1 + \cos\theta_2\right)\hat{r} + \left(\sin\theta_1 - \sin\theta_2\right)\hat{z}\right\}$$

$$\theta_i = 45^\circ \qquad r_i = \frac{a}{2}$$



$$\vec{E}_1 = \frac{-3\lambda}{4\pi\varepsilon_0 (a/2)}\left(\cos 45^\circ + \cos 45^\circ\right)(-\hat{z})$$

$$= \frac{3\sqrt{2}\,\lambda}{2\pi\varepsilon_0 a}\,\hat{z}$$

$$\vec{E}_2 = \frac{2\sqrt{2}\,\lambda}{2\pi\varepsilon_0 a}(-\hat{x})$$

$$\vec{E}_3 = \frac{2\sqrt{2}\,\lambda}{2\pi\varepsilon_0 a}\,\hat{z} \qquad \vec{E}_4 = \frac{\sqrt{2}\,\lambda}{2\pi\varepsilon_0 a}(-\hat{x})$$

$$\longrightarrow \vec{E}_{\text{کل}} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = \frac{\sqrt{2}}{2\pi\varepsilon_0 a}\left(5\hat{z} - 3\hat{x}\right)$$